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## EARLY FOUNDATIONS OF GEOMETRY TEACHING AND LEARNING

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### ABSTRACT

*Geometry is one of the oldest and most fundamental areas of mathematics, and it acts as a connecting link between abstract mathematical thinking and the physical world. The early basis of geometry teaching and learning has a very important role in developing the logical thinking, visualization, and deductive reasoning skills of students. This paper discusses the early basis of geometry teaching and learning, focusing on key definitions, axioms, theorems, and proofs that form the basis of early geometry learning. In addition, the paper discusses the teaching methods, including Euclidean geometry, Van Hiele model of geometric reasoning, spatial cognition, and constructivist models. In addition, the classical theorems, including the Pythagorean Theorem, Triangle Congruence Theorems, and Properties of Parallel Lines, are discussed with their proofs and applications. The paper emphasizes the importance of geometry in mathematical literacy and cognitive development.*

**Keywords:** *Geometry Teaching, Euclid's Axioms, Learning Theories, Proof, Spatial Reasoning*

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### 1. INTRODUCTION

Geometry is a branch of mathematics that is deeply rooted in humanity's endeavors to comprehend, quantify, and describe the physical universe. It is one of the oldest and most fundamental branches of mathematics. From the earliest civilizations, humans have utilized geometric concepts to address practical issues, including the measurement of land, the construction of shelters, the design of instruments, and the navigation of regions. The development of organized geometric knowledge was progressively facilitated by these early applications, which involved the systematic examination of shapes, sizes, and spatial relationships. Geometry has developed from a practical application to a formal and logical discipline, with an emphasis on the properties of points, lines, angles, surfaces, and solids, as well as the relationships between them. This evolution has occurred over time.

Geometry is essential for the interpretation of one's environment, as it offers the means to examine the patterns, structures, and spatial arrangements that are prevalent in daily life. Geometric thinking facilitates a more profound understanding of the environment, whether it is through the analysis of

natural formations, the design of artifacts, or the layout of buildings. In addition, geometry is a critical element of mathematical education, as it facilitates the development of logical reasoning, visualization skills, and problem-solving abilities. Geometry not only reflects humanity's historical endeavors to comprehend the physical world, but it also continues to function as a conduit between real-world experiences and abstract mathematical comprehension. The word geometry comes from two Greek words:

$$geo = \text{earth}, metron = \text{measurement}$$

Therefore, geometry literally means “measurement of the earth.” This is because the definition of geometry is based on its earliest applications, which involved measuring the earth to build structures and establish boundaries.

Historically, geometry was developed in ancient civilizations such as Egypt, Mesopotamia, and Babylon. For example, the Egyptians applied geometric concepts to re-establish the boundaries of fields after the annual flooding of the Nile River. Similarly, the Mesopotamians developed rules for calculating the areas of simple shapes such as rectangles and triangles.

For example, the area of a rectangle was expressed as:

$$A = l \times w$$

where

$l$  represents the length and  $w$  represents the width.

Likewise, the area of a triangle was calculated as:

$$A = \frac{1}{2}bh$$

where

$b$  is the base and  $h$  is the height.

These early formulas illustrate how geometry initially arose from real-world necessity, long before it became a formal deductive science.

The systematic development of geometry attained its first major milestone in the works of Euclid, who lived around 300 BCE. Euclid's famous book "Elements" represented geometric knowledge in the form of an axiomatic system, which signaled the end of geometry as a mere measurement system and the beginning of geometry as a mathematical discipline.

One of the best examples of Euclidean structure is the formula for the equation of a straight line in coordinate geometry:

$$y = mx + c$$

where

$m$  is the slope of the line and  $c$  is the y-intercept.

Such expressions demonstrate how geometry has evolved into both a visual and algebraic framework.

## 2. LITERATURE REVIEW

**Clements et al. (2018)** investigated the teaching and learning of geometry with a great emphasis on its early roots. The authors described how young students developed geometric concepts through informal experiences with shapes, spatial relationships, and patterns in their environment. The authors emphasized that the early teaching of geometry played an essential role in facilitating the later learning of mathematics, especially in the aspects of visualization, reasoning, and problem-solving. The authors' study underscored the significance of developmentally appropriate teaching practices, such as hands-on learning, to enhance children's early foundations in geometry concepts.

**Jones and Tzekaki (2016)** examined a variety of studies on the teaching and learning of geometry in mathematics education, emphasizing how psychological and cognitive viewpoints have had a significant impact on the evolution of geometry instruction. It was highlighted that the development of higher-order thinking skills requires the integration of critical processes such as visualization, logical reasoning, and the production of geometric proofs. Since geometry allows students to go beyond procedural knowledge to deeper conceptual comprehension, the authors argued that it is an essential subject for fostering mathematical thinking. The review also looked at a variety of instructional strategies employed at different educational levels, such as inquiry-based learning, traditional methods, and the use of technology and visual aids. The study also emphasized how crucial it is to match geometry training with kids' cognitive development, especially when it comes to helping them go from an intuitive and informal understanding of shapes to formal, deductive reasoning and proof. It came to

the conclusion that improving students' comprehension, reasoning skills, and general mathematical ability requires a methodical and cognitively focused approach to teaching geometry.

**Borsuk (2018)** explored the theoretical and mathematical foundations of geometry, presenting it as an axiomatic science that is founded on formal definitions, postulates, and logically deduced theorems. It underscored that geometry is not solely the examination of shapes, but a structured system of reasoning in which all concepts are derived from basic assumptions through logical progression. The book emphasized Euclidean geometry as the classical paradigm of deductive structure, illustrating how it offered a systematic framework for comprehending mathematical proof, shape, and space. It also demonstrated how the axiomatic approach bolstered analytical thinking by fostering precision, clarity, and rigorous reasoning skills. Borsuk's work facilitated the recognition of geometry as a rigorous mathematical discipline and a solid foundation for advanced studies in mathematics by presenting it in this formal manner

**Drigas, Kokkalia, and Lytras (2015)** investigated the use of mobile and multimedia learning in preschool education and discovered that using digital technologies greatly improved young children's educational experiences. According to the study, children's comprehension of fundamental concepts, particularly early mathematics and spatial skills, was enhanced by mobile devices and multimedia technologies that facilitated interactive, captivating, and multisensory learning. Children were able to learn at their own pace thanks to these tools, which were found to encourage motivation, active engagement, and customized learning. The authors recommended integrating mobile and multimedia learning environments into early childhood education to enhance teaching efficacy and learning outcomes after concluding that their successful utilization could have a positive impact on cognitive development.

### **3. FUNDAMENTAL DEFINITIONS IN EARLY GEOMETRY**

Early geometry begins with primitive concepts.

#### **Definition 1: Point**

A **point** has no dimension.

$P$

It represents only position.

#### **Definition 2: Line**

A **line** is a set of infinitely many points extending without end.

$$\overleftrightarrow{AB}$$

Equation of a line in coordinate geometry:

$$y = mx + c$$

Where

$m$  = slope,

$c$  = y-intercept.

### **Definition 3: Plane**

A **plane** is a flat surface extending infinitely in all directions.

Example equation:

$$ax + by + cz = d$$

### **Definition 4: Angle**

An angle is formed by two rays with a common endpoint.

$$\angle ABC$$

Measured in degrees or radians.

### **Definition 5: Triangle**

A triangle is a polygon with three sides.

Sum of angles:

$$\angle A + \angle B + \angle C = 180^\circ$$

## **4. EUCLID'S POSTULATES: FOUNDATIONS OF DEDUCTIVE GEOMETRY**

Euclid's geometry begins with five postulates:

### **Postulate 1**

A straight line can be drawn joining any two points.

### **Postulate 2**

A line segment can be extended indefinitely.

**Postulate 3**

A circle may be drawn with any center and radius.

Equation of a circle:

$$(x - a)^2 + (y - b)^2 = r^2$$

**Postulate 4**

All right angles are equal.

**Postulate 5 (Parallel Postulate)**

Through a point not on a given line, exactly one parallel line can be drawn.

**5. EARLY LEARNING OF GEOMETRY: PEDAGOGICAL FOUNDATIONS**

The early learning of geometry is one of the most significant foundations of a child's mathematical education. Geometry is more than just the study of shapes and figures; it is a subject that develops awareness, logical thinking, and the ability to view the world in terms of structure and relationships. In the early years, children begin to learn geometric concepts in an intuitive way as they identify objects, compare shapes, and explore the patterns that exist in their world. The development of geometric knowledge, however, does not happen quickly; instead, it develops gradually through cognitive development and learning.

As a result, educational psychologists and mathematics educators have stressed that learning geometry must be facilitated within appropriate educational frameworks. Among the most influential theories in this respect are Piaget's constructivist theory of cognitive development and the Van Hiele model of geometric thought, which describe how students progress from simple visual recognition of geometric shapes to more sophisticated levels of analytical thinking and formal deductive proof. In addition to these theories, spatial reasoning, encompassing visualization, mental rotation, and geometric transformation, has played a pivotal role in helping students construct meaningful and lasting geometric concepts. Consequently, successful geometry instruction during the early years must be carefully sequenced in accordance with students' cognitive development and must facilitate the gradual development of abstract mathematical reasoning.

### 5.1 Constructivist Foundations (Piaget)

Piaget argued children build geometric understanding gradually:

- Sensorimotor stage: shape recognition
- Concrete stage: classification
- Formal stage: deductive proof

### 5.2 Van Hiele Theory of Geometric Thinking

The Van Hiele model describes levels of geometric understanding:

| Level | Description                        |
|-------|------------------------------------|
| 0     | Visualization (recognizing shapes) |
| 1     | Analysis (properties noticed)      |
| 2     | Informal Deduction (relationships) |
| 3     | Formal Deduction (proof)           |
| 4     | Rigor (axiomatic reasoning)        |

Teaching must follow this developmental sequence.

### 5.3 Spatial Reasoning in Geometry Learning

Spatial cognition includes:

- Mental rotation
- Visualization
- Geometric transformations

Example transformation:

$$(x, y) \rightarrow (-x, y)$$

Reflection across y-axis.

## 6. MAJOR THEOREMS IN EARLY GEOMETRY WITH PROOFS

The evolution of geometry as a deductive and logical area of mathematics is essentially based on its theorems, which are established relationships between geometric entities like points, lines, angles, and shapes. When students are initially learning geometry, the presentation of key theorems is an important aspect of enabling them to progress from basic visualization concepts to advanced mathematical concepts. These theorems not only act as important problem-solving tools in geometry but also as building blocks for developing logical thinking and deduction skills.

Classical examples like the angle sum theorem for triangles, the Pythagorean theorem, criteria for triangle congruence, and the corresponding angles theorem are the foundation of elementary geometry education globally. By using rigorous proofs, students come to understand that geometric facts are not merely observationally based but are instead developed from axioms and postulates using rigorous logic. Therefore, the exploration of these early theorems provides students with not only problem-solving skills but also an introduction to the formalism of mathematics as a proof-based science.

### 6.1 Theorem 1: Angle Sum Property of a Triangle

#### Statement

The sum of interior angles of any triangle is:

$$180^\circ$$

#### Proof

Let triangle  $ABC$ .

Draw a line through  $A$  parallel to  $BC$ .

Then alternate interior angles give:

$$\angle ABC = \angle 1, \angle ACB = \angle 2$$

Straight line angle at  $A$ :

$$\angle 1 + \angle A + \angle 2 = 180^\circ$$

Thus:

$$\angle A + \angle B + \angle C = 180^\circ$$

Proved.

### Example

If two angles are  $60^\circ$  and  $80^\circ$ :

$$\angle C = 180^\circ - (60^\circ + 80^\circ) = 40^\circ$$

## 6.2 Theorem 2: Pythagorean Theorem

### Statement

In a right triangle:

$$a^2 + b^2 = c^2$$

where  $c$  is the hypotenuse.

### Proof (Area Method)

Construct a square of side  $a + b$ . Inside it place four right triangles.

Total area:

$$(a + b)^2 = a^2 + b^2 + 2ab$$

Area inside also equals:

$$4\left(\frac{1}{2}ab\right) + c^2 = 2ab + c^2$$

Equating:

$$a^2 + b^2 + 2ab = 2ab + c^2$$

Thus:

$$a^2 + b^2 = c^2$$

Proved.

### Example

Given  $a = 3$ ,  $b = 4$ :

$$c = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$$

## 6.3 Theorem 3: Congruence of Triangles (SSS)

**Statement**

If three sides of one triangle equal three sides of another, the triangles are congruent.

$$AB = DE, BC = EF, CA = FD$$

Then:

$$\triangle ABC \cong \triangle DEF$$

**Proof (Rigid Motion Argument)**

One triangle can be translated and rotated until sides overlap perfectly, forcing full coincidence.

Thus:

$$\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$$

Proved.

**6.4 Theorem 4: Corresponding Angles Theorem****Statement**

If a transversal cuts two parallel lines, corresponding angles are equal.

$$l_1 \parallel l_2 \Rightarrow \angle 1 = \angle 2$$

**Application Example**

If  $\angle 1 = 110^\circ$ , then:

$$\angle 2 = 110^\circ$$

**7. GEOMETRY TEACHING APPROACHES IN EARLY EDUCATION**

Geometry is a special topic in early mathematics education because it bridges the world of visual experience with the world of abstract thinking. Unlike arithmetic, which typically starts with numbers and symbols, geometry starts with shapes, space, and patterns that children experience in the world around them. But successful geometry teaching goes beyond simply presenting definitions and

formulas. Successful geometry teaching must help students develop their spatial reasoning, conceptual understanding, and deductive thinking skills over time. Researchers in mathematics education have stressed that early geometry learning should be active, developmental, and logically organized. There are several geometry teaching strategies that have been found to be very effective in building young students' foundational geometric knowledge.

### 7.1 Concrete to Abstract Learning

Among the most widely suggested approaches in early geometric instruction is the concrete-abstract approach. This is because geometric ideas are often very visual and sometimes hard to conceptualize mentally. Students would benefit from the concrete-abstract approach in that they would be introduced to geometric ideas through concrete experiences before progressing to abstract thoughts.

#### ➤ Concrete Stage: Manipulatives and Physical Models

At the earliest stage, children explore geometry through hands-on materials such as:

- Building blocks
- Pattern tiles
- Geoboards
- Paper folding activities
- 3D models of solids

For example, a child can physically construct a cube using blocks, helping them understand faces, edges, and vertices.

Cube has 6 faces, 12 edge, 8 vertices

Such tactile learning strengthens spatial awareness and shape recognition.

#### ➤ Representational Stage: Visual Drawings and Diagrams

After working with objects, students move on to creating geometric figures and analyzing diagrams. This phase fills the gap between the physical world and mathematical concepts.

For example, students can create triangles and analyze properties like:

$$\angle A + \angle B + \angle C = 180^\circ$$

Through repeated visual representation, learners begin recognizing geometric relationships.

## 7.2 Inquiry-Based Geometry

Another effective method in the early education of geometry is inquiry-based learning. In this way, students are encouraged to actively discover properties in geometry instead of merely memorizing them.

Rather than presenting a theorem to students, teachers are encouraged to help them discover patterns, ask questions, and draw conclusions based on investigation.

### ➤ Example: Discovering Triangle Angle Sum

Students can measure angles in different triangles using protractors:

- Triangle 1:  $50^\circ, 60^\circ, 70^\circ$
- Triangle 2:  $90^\circ, 40^\circ, 50^\circ$

By adding the angles, students observe:

$$50 + 60 + 70 = 180^\circ$$

$$90 + 40 + 50 = 180^\circ$$

Thus, learners discover empirically that:

$$\angle A + \angle B + \angle C = 180^\circ$$

Such inquiry fosters curiosity and deeper conceptual understanding.

## 7.3 Proof as a Foundation of Reasoning

One of the defining features of geometry is its focus on proof, which makes geometry one of the first places in mathematics where students are exposed to formal logical argument.

Proof is critical because it shows that the statements of geometry are not observed but logically derived from axioms and postulates.

For instance, the Pythagorean theorem is not just a rule but a proven statement:

$$a^2 + b^2 = c^2$$

## 8. CONCLUSION

Geometry, as one of the oldest areas of mathematics, has greatly influenced the human understanding of space and logical reasoning. From its application in ancient civilizations to its development using Euclid's axiomatic method, geometry has become a deductive area of mathematics. This paper has discussed some of the fundamental concepts of early geometry education, including fundamental ideas, necessary postulates, and traditional theorems that teach students about proof and reasoning. It has also discussed the importance of cognitive development theories like Piaget's stages and the Van Hiele model of understanding how students develop geometric concepts. Spatial reasoning and teaching strategies like concrete-abstract, inquiry-based, and step-by-step proof teaching have been demonstrated to play a vital role in effective learning. Fundamentals of geometry are critical for mathematical literacy and developing critical thinking skills necessary in advanced mathematics and problem-solving.

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